Bounding slopes of p-adic modular forms

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Abstract

Let p be prime, N be a positive integer prime to p, and k be an integer. Let $P_k(t)$ be the characteristic series for Atkin's U operator as an endomorphism of p-adic overconvergent modular forms of tame level N and weight k. Motivated by conjectures of Gouvêa and Mazur, we strengthen a congruence in [W] between coefficients of P_k and $P_{k'}$ for k' p-adically close to k. For $p-1 \mid 12$, $N=1$, $k=0$, we compute a matrix for U whose entries are coefficients in the power series of a rational function of two variables. We apply this computation to show for $p=3$ a parabola below the Newton polygon $\mathbf{N}_0$ of P_0, which coincides with $\mathbf{N}_0$ infinitely often. As a consequence, we find a polygonal curve *above* $\mathbf{N}_0$. This tightest bound on $\mathbf{N}_0$ yields the strongest congruences between coefficients of P_0 and P_k for k of large 3-adic valuation.

1 Overview and background

Let p be a prime number, N be a positive integer relatively prime to p, and k be an integer. Let B be a p-adic ring between $\mathbf{Z}_p$ and $\emptyset_p$, the ring of integers in $\mathbf{C}_p$. Denote by $\mathcal{M}_k(N,B)$ the p-adic overconvergent modular forms of tame level N and weight k and by $\mathcal{S}_k(N,B)$ the subspace of overconvergent cusp forms.

For every weight k, Atkin's U operator is an endomorphism of $\mathcal{M}_k(N,B)$ stabilizing $\mathcal{S}_k(N,B)$. Denote by $U^{(k)}$ the restriction of U to $\mathcal{M}_k(N,B)$ and by $U_{(k)}$ the restriction to $\mathcal{S}_k(N,B)$. These are compact operators, so the characteristic series

$$P_k(t) = \det(1 - tU^{(k)}), \quad Q_k(t) = \det(1 - tU_{(k)})$$

exist.

Let $a_m(P_k)$ be the coefficient of t^m in $P_k(t)$. As a function on a suitably defined space of weights k, $a_m(P_k)$ is a rigid analytic function of k.

Wan[W], and Buzzard[B] construct $\hat{\mathbf{N}}(m)$, which grows as $O(m^2)$ and depends on p and N and not on k such that $v_p(a_m(P_k)) > \hat{\mathbf{N}}(m)$.

Gouvêa and Mazur[GM] note, in an earlier work, the existence of $\hat{\mathbf{N}}(m)$ and show, for prime $p \geq 5$, integer l and positive integer n,

$$v_p(a_m(P_k) - a_m(P_{k+lp^n(p-1)})) \geq n+1. \tag{1}$$

Following a remark in [Ka], the result in Equation (1) extends to $p = 2, 3$.

In section 2, we show

$$v_p(a_m(P_k) - a_m(P_{k+lp^n(p-1)})) \geq \hat{\mathbf{N}}(m-2) + n + 1. \tag{2}$$

In section 3, for each $p = 2, 3, 5, 7, 13$, $N = 1$, we construct a matrix M for $U_{(0)}$ with respect to an explicit basis. We show, for M_{ij} the entries of M,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} M_{ij} x^i y^j$$

is the power series expansion of a rational function of two variables.

In section 4, we show for $p = 3$,

$$v_p(a_m(Q_0)) \geq 3\binom{m}{2} + 2m,$$

with equality if and only if there is positive integer j such that $m = (3^j - 1)/2$. The secant segments joining these vertices of the Newton polygon $\mathbf{N}_0'$ of Q_0 form a polygonal curve *above* $\mathbf{N}_0'$. We find evidence in support of a conjecture in [G] on the distribution of slopes of classical modular forms.

1.1 Motivating conjectures

The zeros of $P_k(t)$ are reciprocals of $U^{(k)}$ eigenvalues. For rational number α, let $d(k, \alpha)$ denote the number of $U^{(k)}$ eigenvalues with p-adic valuation α.

Conjecture 1.1 (Gouvêa-Mazur) *Let k, l be integers, n be a positive integer, and $\alpha < n$. Then $d(k, \alpha) = d(k + lp^n(p-1), \alpha)$.*

Wan [W] uses Equation (1) and the construction of $\hat{\mathbf{N}}(m)$ to compute a quadratic concave up function $f_{Wan}(n)$ such that the conclusion of Conjecture 1.1 holds for $\alpha < f_{Wan}(n)$.

The stronger congruence in Equation (2) together with the method of [W] shows there is quadratic $f(n)$ with quadratic term smaller than that of $f_{Wan}(n)$ such that the conclusion of Conjecture 1.1 holds for $\alpha < f(n)$

Conjecture 1.2 (Gouvêa) *Let R_k be the multiset of slopes with multiplicity of classical p-oldforms in $\mathcal{M}_k(N, \mathbf{Z}_p)$. The probability that an element of R_k chosen with uniform distribution is in the interval $(\frac{k-1}{p+1}, \frac{p(k-1)}{p+1})$ diminishes to zero as k increases without bound.*

1.2 Spaces of overconvergent modular forms

For $p \geq 5$, let E_{p-1} be the level one Eisenstein series. Let $M_k(N,B)$ be the classical weight k level N modular forms with coefficients in B.

Proposition 1.3 (Katz) *For $p \geq 5$ and any $f \in \mathcal{M}_k(N,B)$, there are $b_j \in M_{k+j(p-1)}(N,B)$ for $j \geq 0$ and $r \in \emptyset_p$ of positive valuation such that*

$$f = b_0 + \sum_{j=1}^{\infty} r^j b_j / E_{p-1}^j, \tag{3}$$

There is a distinguished choice of b_j after choosing r and direct sum decompositions

$$M_{k+j(p-1)} = E_{p-1} \cdot M_{k+(j-1)(p-1)} \oplus W_{k+j(p-1)},$$

such that $b_j \in W_{k+j(p-1)}(N,B)$ for $j > 0$.

See [Ka], Propositions 2.6.1 and 2.8.1. The parameter r is the *growth condition* and $v_p(r)$ is bounded above by the given f.

Let $\mathcal{M}_k(N,B,r)$ be the space of modular forms with growth condition r. The space $\mathcal{M}_k(N,B)$ is $\bigcup_{v_p(r)>0} \mathcal{M}_k(N,B,r)$.

Remark 1.3.1 *For $p = 3$, $N > 2$ and prime to 3, Theorem 1.7.1 of [Ka] shows there is a level N lift of the characteristic 3 Hasse invariant, so an analogous expansion result holds. Proposition 2.8.2 of* loc. cit. *shows the expansion result for $N = 2$.*

Proposition 1.4 *Suppose $p = 2$ or 3 and N relatively prime to p. For any $f \in \mathcal{M}_k(N,B)$ there are $b_j \in M_{k+4j}(N,B)$ and $r \in \emptyset_p$ of positive valuation such that*

$$f = b_0 + \sum_{j=1}^{\infty} r^{4j/(p-1)} b_j / E_4^j,$$

There is a distinguished choice of b_j after choosing r and direct sum decompositions

$$M_{k+4j} = E_4 \cdot M_{k+4(j-1)} \oplus W_{k+4j}(N,B),$$

such that $b_j \in W_{k+4j}(N,B)$ for $j > 0$.

Proof. We follow the remark at the end of Subsection 2.1 of *loc. cit.*. Let B be the fourth power of the Hasse invariant A for $p = 2$ and the square of A for $p = 3$. In either case, B is a weight 4 level 1 modular form defined over $\mathbf{F}_p$. A version of Deligne's congruence holds: $B \equiv E_4 \bmod 2^4$ and $B \equiv E_4 \bmod 3$.

For $N > 2$, and relatively prime to p, the functor "isomorphism classes of elliptic curves with level N structure" is representable by a scheme which is smooth over $\mathbf{Z}[\frac{1}{N}]$ and the formation of modular forms commutes with base change to a ring in which p is topologically nilpotent. So we repeat the construction of p-adic modular forms for $p = 2, 3$ and Katz expansions with powers of $r^{4/(p-1)} E_4^{-1}$.

For $p = 2, 3$ (and 5), and $N = 1$, Section 1.4 of [Se2] states weight zero forms have expansions in powers of ΔE_4^{-3} where Δ is the weight 12 level 1 cusp form. Coleman[C2] shows

$$E_k \cdot \mathcal{M}_0(N, B) = \mathcal{M}_k(N, B).$$

$M_k(N, B)$ is a free B module, so $M_k(N, B) = E_4 \cdot M_{k-4}(N, B) \oplus W_k(N, B)$ for some $W_k(N, B) \subset M_k(N, B)$. □

Theorem 1.5 (Coleman) *Let k_1, k_2 be weights. Let $G(q) \in M_{k_1 - k_2}(N, B)$. Let Ξ be the operator multiplication by $G(q)/G(q^p)$. If $1/G \in \mathcal{M}_{k_2 - k_1}(N, B)$ then $U^{(k_1)}$ is similar to $U^{(k_2)}\Xi$.*

Remark 1.5.1 *The Eisenstein series satisfy the hypothesis of Theorem 1.5.*

1.3 Notations for matrices and Newton Polygons

Let M be a matrix over a ring, possibly of infinite rank. Let n be a nonnegative integer. Let $s = (s_1, s_2, s_3, ...s_n)$ be a sequence of n distinct natural numbers.

The $n \times n$ *diagonal major* of M associated to s is the $n \times n$ matrix A whose entry A_{ij} is M_{s_i, s_j}.

A *selection* of a M associated to s and degree n permutation π is a sequence of n elements, $(M_{s_1, s_{\pi(1)}}, M_{s_2, s_{\pi(2)}}, \ldots, M_{s_n, s_{\pi(n)}})$.

The $n \times n$ *diagonal minor* of M associated to s is the determinant of the $n \times n$ diagonal major of M associated to s.

The *upper $n \times n$ diagonal major* of M is the diagonal major associated to the sequence $(1, 2, 3, \ldots n)$.

The diagonal matrix $D = diag(d_i : i \geq 1)$ is the matrix with entries $D_{ii} = d_i$ and zero elsewhere.

The Newton polygon of power series $P(t)$ is the function $\mathbf{N}(m)$ which is the lower convex hull of the set $(m, v_p(a_m(P)))$, defined for real $m \geq 0$.

A vertex of the Newton polygon $\mathbf{N}(m)$ is a point $(m, \mathbf{N}(m))$ such that $\mathbf{N}(m) = v_p(a_m(P))$.

A side of a Newton polygon $\mathbf{N}(m)$ is a line segment whose endpoints are vertices.

The slopes of a Newton polygon are the slopes of its sides.

The multiplicity of a slope is the difference of the first coordinates of its endpoints.

We denote by $\mathbf{N}_k(m)$ the Newton polygon of P_k, and by $\hat{\mathbf{N}}_k(m)$ a function such that $\mathbf{N}_k(m) \geq \hat{\mathbf{N}}_k(m)$. We indicate by $\hat{\mathbf{N}}(m)$ a function such that for all weights k, $\mathbf{N}_k(m) \geq \hat{\mathbf{N}}(m)$.

We denote by $\mathbf{N}'_k(m)$ the Newton polygon of Q_k, and by $\hat{\mathbf{N}}'_k(m)$ a function such that $\mathbf{N}'_k(m) \geq \hat{\mathbf{N}}'_k(m)$.

We state as Lemma 3.2 that if $p - 1 \mid 12$ and $N = 1$, then $P_k(t) = (1 - t)Q_k(t)$. For these cases, $\mathbf{N}'_k(m) = \mathbf{N}_k(m + 1)$.

2 Comparing Newton polygons for U in different weights

Retain p, N, k as before, and let l be an integer and n be a positive integer. For $p = 2$, we require $n \geq 2$. Let $k' = k + l(p-1)p^n$. At the end of the section, we show there is a quadratic $\hat{\mathbf{N}}(m)$ such that

$$v_p(a_m(P_k) - a_m(P_{k'})) \geq \hat{\mathbf{N}}(m-2) + n + 1.$$

We now describe only the case $p > 3$ for clarity. Section 1.2 reviews the differences for $p = 2, 3$ from the case $p > 3$.

Let $r = p^{1/(p+1)}$. Choose a basis $\{b_{0,s}\}$ for the module $M_k(N, B)$. For $i > 0$, choose a basis $b_{i,s}$ for the module $W_{k+i(p-1)}(N, B)$.

Let $e_{i,s} = r^i E_{p-1}^{-i} b_{i,s}$. Let M be the matrix for $U^{(k)}$ with respect to the basis $\{e_{i,s}\}$.

Let $\mathbf{N}_k(m)$ be the Newton polygon of $P_k(t)$.

Lemma 3.1 of [W] includes

Lemma 2.1 *Let $M_{i,s}^{u,v}$ be the coefficient of $e_{u,v}$ in $U^{(k)}(e_{i,s})$.*
Then $v_p(M_{i,s}^{u,v}) \geq u(p-1)/(p+1)$.

Let $d_u = \dim M_{k+u(p-1)}(N, B) \otimes \mathbf{C}_p$. For $u > 0$, let $m_u = d_u - d_{u-1}$.

Lemma 2.2 (Wan) *Let k be a weight. If $d_v \leq m < d_{v+1}$ for some $v \geq 0$, then*

$$\mathbf{N}_k(m) \geq \frac{p-1}{p+1}\left(\sum_{u=0}^{v} u m_u + (v+1)(m - d_v)\right) - m. \tag{4}$$

Definition 2.1 *Let $\hat{\mathbf{N}}_k(m)$ be the right side of Equation (4).*

The m_u have an upper bound, depending on p and N, so $\hat{\mathbf{N}}_k(m)$ grows quadratically.

Wan shows $\mathbf{N}_k(m) = \mathbf{N}_{k'}(m)$ when both are less than $n + 1$.

Lemma 2.3 *Let A be the matrix for $U^{(k)}$ with respect to basis $\epsilon_{i,s} = r^{-i} e_{i,s}$. Then*

$$v_p(A_{i,s}^{u,v}) \geq (up - i)/(p+1)$$

and also at least zero.

Proof. $U^{(k)}$ stabilizes $\mathcal{M}_k(N, B, 1)$, as shown in [GM]. □

Proposition 2.4 $E_{p-1}^{p^n}(q)/E_{p-1}^{p^n}(q^p) \in 1 + p^{n+1}\mathcal{M}_0(1, \mathbf{Z}_p, 1)$.

Proof. In weight zero, the only $\epsilon_{i,s}$ not 0 at the cusp $\underline{\infty}$ is the constant function 1. The q-expansion of $(E_{p-1} - 1)/p$ is in $q\mathbf{Z}[[q]]$. □

Theorem 2.5 *For k, k' as above, $v_p(a_m(P_k) - a_m(P_{k'})) \geq \hat{\mathbf{N}}_k(m-2) + n + 1$.*

Proof. Let C be the matrix with respect to the basis $\epsilon_{i,s}$ for multiplication by $E_{p-1}^{p^n}(q)/E_{p-1}^{p^n}(q^p)$ considered as an operator on $\mathcal{M}_k(N,B,r)$.

Let $M^{(k')} = MC$. By Theorem 1.5, $M^{(k')}$ is a matrix for an operator similar to $U^{(k')}$ on $\mathcal{M}_{k'}(N,B,r)$ and $M^{(k')}$ acts on $\mathcal{M}_k(N,B,r)$.

By Proposition 2.4, the matrix $C-1$ is a matrix with entries in $p^{n+1}B$, so $M-M^{(k')}$ has entries in $p^{n+1}B$.

The difference $a_m(P_k) - a_m(P_{k'})$ is equal to

$$\mathrm{tr} \bigwedge^m M - \mathrm{tr} \bigwedge^m M^{(k')}.$$

These traces are the sums of all the different $m \times m$ diagonal minors of M and $M^{(k')}$, so the difference contains terms (up to sign)

$$\prod_{i=1}^m M^{(k')}_{s_i,s_{\pi(i)}} - \prod_{i=1}^m M_{s_i,s_{\pi(i)}}, \tag{5}$$

where s is a sequence of m integers, π is a permutation of degree m.

Let

$$Z = \prod_{i=1}^m (z_i + w_i) - \prod_{i=1}^m (z_i), \tag{6}$$

where $z_i \in B$ and $w_i \in p^{n+1}B$, be instance of equation (5).

The sequence $(z_1, z_2, ...z_m)$ is a selection of M. By Lemma 2.3, the product of any $m-j$ of them has valuation at least $\hat{\mathbf{N}}_k(m-2j)$. The product of any j of the w_i has valuation at least $j(n+1)$.

Rewrite (6) as

$$Z = \sum_{\emptyset \neq s \subset \{1,2,...m\}} \prod_{i \in s} w_i \prod_{i \notin s} z_i. \tag{7}$$

For any subset s of size j,

$$v_p(\prod_{i \in s} w_i \prod_{i \notin s} z_i) \geq \hat{\mathbf{N}}_k(m-2j) + j(n+1).$$

The set s is nonempty, so,

$$v_p(Z) \geq \hat{\mathbf{N}}_k(m-2) + n + 1,$$

for every instance of Equation (6).

□

Corollary 2.5.1 *There is a quadratic $\hat{\mathbf{N}}(m)$ independent of k such that the conclusion of Theorem 2.5 holds.*

Proof. Given p, N, Wan[W] shows there are finitely many different $\hat{\mathbf{N}}_k(m)$. Let $\hat{\mathbf{N}}(m)$ be the infimum of them. □

3 Computing tame level 1 U for $p \in \{2, 3, 5, 7, 13\}$

Let p be a prime such that $X_0(p)$ has genus 0, that is, $p \in \{2, 3, 5, 7, 13\}$ and $N = 1$. We show how to compute $U_{(0)}$ with respect to an explicit basis.

The curve $X_0(p)$ has a uniformizer

$$d_p = \sqrt[p-1]{\Delta(q^p)/\Delta(q)}$$

with simple zero at the cusp $\underline{\infty}$, pole at the cusp $\underline{0}$, and leading q expansion coefficient 1.

Let $\pi : X_0(p) \to X_0(1)$ be the map which ignores level p structure. Let $\hat{\jmath} = \pi^*(j)$. The map π is ramified above $j = 0, 1728, \infty$ only.

Proposition 3.1 *There is a degree $p+1$ polynomial H_p over $\mathbf{Z}$ with constant term 1 such that*

$$d_p \hat{\jmath} = H_p(d_p).$$

PROOF. The map π has degree $p + 1$. The product $d_p \hat{\jmath}$ has a pole only at the cusp $\underline{0}$. Hence, there is a polynomial H_p satisfying the proposition.

H_p has integer coefficients, because the q-expansion of $d_p \hat{\jmath}$ at $\underline{\infty}$ is in $1 + q\mathbf{Z}[[q]]$ $\square$

Remark 3.1.1 *The ramification degrees of π over $j = 0$ are 1 and 3, yielding roots of multiplicity 1 or 3 of $H_p(d_p)$. Points over $j = 1728$ are roots of multiplicity 1 or 2 of $H_p(d_p) - 1728d_p$. We calculate H_p by equating q-expansions.*

Lemma 3.2 $P_k(t) = (1 - t)Q_k(t)$

PROOF. $X_0(p)$ has genus 0, so the only weight zero noncuspidal eigenforms are constants and the eigenvalue is 1. By a theorem of [H], or as a consequence of Theorem 1.5, in every weight k, $d(k, 0) = 1$ and a slope zero eigenform is noncuspidal. $\square$

Let $t_2 = 4$, $t_3 = 3$. For $p \geq 5$, let $t_p = 1$.

Let $c_2 = 0$, $c_3 = 1728$, $c_5 = 0$, $c_7 = 1728$, and $c_{13} = 432000/691$.

Let $e = 12/(p^2 - 1)$.

Lemma 3.3 *The Newton polygon of $H_p(d_p) - c_p d_p$, as a polynomial in d_p, has a single side of slope ep.*

Lemma 3.4 *The weight 12 power of $E_{t_p(p-1)}$ is $(j - c_p)\Delta$.*

The lemmas are direct computations.

Proposition 3.5 *Let $r < p/(p+1)$. The disc $D = \{z : z \in X_0(1), v_p(E_{t_p(p-1)}(z)) < t_p r\}$ is isomorphic to $\{z : z \in X_0(p), v_p(d_p(z)) > -er(p+1)\}$.*

Proof. When $z \in X_0(1)$ is a point of supersingular reduction, $\Delta(z)$ is a unit. At a point of ordinary reduction, $E_{t_p(p-1)}(z)$ is a unit and $v_p(\Delta(z)) \geq 0$. By Lemma 3.4, $D = \{z : v_p(j(z) - c_p) < er(p+1)\}$.

Lemma 3.3 shows the relation $(\hat{\jmath} - c_p)d_p = H_p - c_p d_p$ is uniquely invertible for d_p such that $v_p(d_p(z)) > -er(p+1)$, establishing the isomorphism. □

Corollary 3.5.1 $\mathcal{S}_0(1, \mathbf{Z}_p) \subset d_p\mathbf{Z}_p[[d_p]]$. *$U_{(0)}$ acts as a matrix M on a basis of powers of d_p.*

Let $\mathcal{W}$ be the rigid subspace of $X_0(p)$ where $v_p(\pi^*(E_{t_p(p-1)})) < t_p/(p+1)$. The section s of π over $\pi(\mathcal{W})$ such that for elliptic curve E, $s(E)$ is the pair (E, C) for C the canonical order p subgroup of E is an isomorphism.

Let V be the pullback of ϕ, the Deligne-Tate lift of Frobenius on $X_0(1)/\mathbf{F}_p$. Let w_p be the Atkin-Lehner involution on $X_0(p)$.

Lemma 3.6 *For points of $\mathcal{W}$,*

$$V(j) \circ \pi = \hat{\jmath} \circ w_p.$$

Proof. The Atkin-Lehner involution acts as

$$w_p : (E, C) \to (E/C, E[p]/C).$$

E has a canonical subgroup of order p, and

$$V : E \to E/\ker\phi^*,$$

coincides with $s^* \circ w_p^* \circ \pi^*$. □

We identify $\mathcal{W}$ with $\pi(W)$ via section s.

Proposition 3.7 *For points of $\mathcal{W}$,*

$$H_p(p^{12/(1-p)}/d_p)V(d_p) - p^{12/(1-p)}H_p(V(d_p)/d_p = 0. \tag{8}$$

Proof. The modular equation

$$H_p(w_p^*(d_p))V(d_p) = H_p(V(d_p))w_p^*(d_p)$$

holds on $\mathcal{W}$ and $w_p(d_p) = (p^{12/(1-p)}/d_p)$. □

Theorem 3.8 *There is an algebraic function $I_p(y, x)$ and a matrix M for $U_{(0)}$ with respect to the basis d_p^n such that entries M_{ij} satisfy a generating function equation*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} M_{ij}x^i y^j = \frac{y}{p}\frac{d}{dy}\log I_p(x, y). \tag{9}$$

PROOF. Clear denominators and factor $V(d_p)-w_p^*(d_p)$ from Equation (8) to determine an algebraic relation

$$d_p^p I_p(V(d_p), 1/d_p)$$

between d_p and $V(d_p)$, of degree p in d_p. The inverse of V applied to coefficient of d_p^{p-1} is $\mathrm{t}r\ V(d_p) = pU(d_p)$.

The values of $U(d_p^n)$ for $n=0$ to $p-1$ and the coefficients of I_p determine a recurrence for $U(d_p^n)$ for $n \geq p$. $\square$

Remark 3.8.1 *The $I_p(x,y)$ for $p=2,3,5,7,13$ are*

$$\begin{aligned}
I_2 &= 1-(2^{12}x^2+3\cdot 2^4x)y - xy^2,\\
I_3 &= 1-(3^{12}x^3+4\cdot 3^8x^2+10\cdot 3^3x)y-(3^6x^2+4\cdot 3^2x)y^2-xy^3,\\
I_5 &= 1-(5^{12}x^5+6\cdot 5^{10}x^4+63\cdot 5^7x^3+52\cdot 5^5x^2+63\cdot 5^2x)y\\
&\quad -(5^9x^4+6\cdot 5^7x^3+63\cdot 5^4x^2+52\cdot 5^2x)y^2\\
&\quad -(5^6x^3+6\cdot 5^4x^2+63\cdot 5x)y^3-(5^3x^2+6\cdot 5x)y^4-xy^5,\\
I_7 &= 1-(7^{12}x^7+4\cdot 7^{11}x^6+46\cdot 7^9x^5+272\cdot 7^7x^4+\\
&\quad 845\cdot 7^5x^3+176\cdot 7^2x^2+82\cdot 7x)y-\ldots-xy^7,\\
I_{13} &= 1-(13^{12}x^{13}+2\cdot 13^{12}x^{12}+25\cdot 13^{11}x^{11}+196\cdot 13^{10}x^{10}+\\
&\quad 1064\cdot 13^9x^9+4180\cdot 13^8x^8+12086\cdot 13^7x^7+\\
&\quad 25660\cdot 13^6x^6+39182\cdot 13^5x^5+41140\cdot 13^4x^4+\\
&\quad 27272\cdot 13^3x^3+9604\cdot 13^2x^2+1165\cdot 13x)y-\ldots-xy^{13}.
\end{aligned}$$

Proposition 3.9 *The p-adic valuation of M_{ij} is at least $e(pi-j)-1$. There is a parabola $\hat{\mathbf{N}}(m)$ with quadratic coefficient $6/(p+1)$ such that $\mathbf{N}_0(m) \geq \hat{\mathbf{N}}(m)$.*

PROOF. Let $M'_{ij} = p^{e(j-i)}M_{ij}$. The matrix (M'_{ij}) is similar to (M_{ij}). Theorem 3.8 shows

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} M'_{ij}x^iy^j = \frac{y}{p}\frac{d}{dy}\log I_p(p^{-e}x, p^ey). \tag{10}$$

Direct calculation shows $I_p(p^{-e}x, p^ey)$, for the I_p displayed in Remark 3.8.1, is a polynomial in $p^{e(p-1)}x$ and y with integer coefficients. Hence, $v_p(M'_{ij}) \geq i\cdot e(p-1)$. $\square$

3.1 Tame level 1 and $p=2$ or 3

Emerton[E] calculates the lowest positive slope 2-adic modular forms of every weight. Concise expressions for the q-expansions of a few forms facilitate computation.

Serre[Se] observes that for a compact operator M expressed as a matrix on a basis of a Banach space, if c_i is the infimum of the valuations of column i of M, then $\mathrm{t}r\ (\wedge^n M)$ has valuation at least the sum of the n smallest c_i.

Proposition 3.10 *For $p = 2$ and even weight k, there is an $\emptyset_2$ basis $\{e_n\}_{n\geq 1}$ of $\mathcal{S}_k(1, \emptyset_2)$ such that the image of $U_{(k)}$ is a subset of $\bigoplus 8^n e_n \emptyset_2$.*

Proof. This is a rewriting of Proposition 3.21 of [E] in language amenable to the noted observation of Serre. The basis element e_n is $F_k d_2^n$ for a certain weight k form F. □

Recall $\mathbf{N}'_k(m)$ is the Newton polygon of $Q_k(t)$.

Corollary 3.10.1 $\mathbf{N}'_k(m) \geq 3\binom{m+1}{2}$.

Lemma 3.11 *Suppose $p = 3$. Let $S = \sqrt[8]{\Delta^3/V(\Delta)}$. S^2 is in $\mathcal{M}_6(1, \mathbf{Z}_p)$ and does not vanish at the cusp $\underline{\infty}$. The quotient $S/V(S)$ is in $M_0(1, \emptyset_3, 3/2)$ and as a power series in $Z[[d_3]]$, $S/V(S) - 1$ is in the ideal $(9d_3, 27d_3^2)$.*

Proof. Direct calculation and comparison of q expansions shows S is the Eisenstein series for level 3, weight 3 and character τ, the 3-adic Teichmuller character. S^2 is a level 3 weight 6 classical modular form and thus a tame level 1 weight 6 overconvergent modular form.

The curve $X_0(9)$ has genus zero and uniformizer

$$d_9 = \sqrt[8]{V(V(\Delta))/\Delta}.$$

The ramification of the forgetful map to $X_0(3)$ shows

$$d_3 = d_9 + 9d_9^2 + 27d_9^3.$$

Reversal of this relation between d_3 and d_9 and the observation

$$S/V(S) = d_9/d_3$$

shows $S/V(S)$ is in $M_0(1, \emptyset_3, 3/2)$, has constant term 1, and $S/V(S) - 1 \in (9d_3, 27d_3) \subset Z[[d_3]]$. □

Proposition 3.12 *For $p = 3$ and even weight k divisible by 3, $\mathbf{N}'_k(m) \geq 3\binom{m}{2}$.*

Proof. Let R be the multiplication by $(S/V(S))^{k/3}$ operator. Theorem 1.5 shows the composition $U_{(0)}R$ is similar to $U_{(k)}$. Lemma 3.11 shows the conclusions of Proposition 3.9 hold for $U_{(0)}R$. □

3.2 Further example for $p = 3,\ N = 1, k = 0$.

Let $p = 3$, $N = 1$ and

$$\hat{\mathbf{N}}'_0(m) = \frac{3}{2}m(m-1) + 2m.$$

We work an example of Proposition 3.9.

Lemma 3.13 $\mathbf{N}'_0(m) \geq \hat{\mathbf{N}}'_0(m)$.

PROOF. Recall $e = 3/2$. Equation (10) shows

$$3\sum_{i,j} M'_{ij}x^iy^j = \frac{9(10xy + 8\sqrt{3}xy^2 + 3xy^3) + 3^5(4\sqrt{3}x^2y + 2x^2y^2) + 3^8x^3y}{1 - 3^3(10xy + 4\sqrt{3}xy^2 + xy^3) - 3^6(4\sqrt{3}x^2y + x^2y^2) - 3^9x^3y}. \quad (11)$$

Following the last step of Proposition 3.9, substitute $\delta = 3^3x$ into the right side of Equation (11) to get

$$G(\delta, y) = \frac{10\delta y + 8\sqrt{3}\delta y^2 + 3\delta y^3 + 4\sqrt{3}\delta^2 y + 2\delta^2y^2 + \delta^3 y}{1 - 10\delta y - 4\sqrt{3}(\delta y^2 + \delta^2 y) - (\delta y^3 + \delta^2y^2 + \delta^3 y)}. \quad (12)$$

The valuation of M'_{ij} is at least $i \cdot e(p-1) - 1 = 3i - 1$. So

$$\mathbf{N}'_0(m) \geq \sum_{i=1}^{m} 3i - 1 = \hat{\mathbf{N}}'_0(m).$$

$\square$

4 For $p = 3$, $N = 1$, $\hat{\mathbf{N}}'_0$ is a sharp parabola below $\mathbf{N}'_0$

Let $p = 3$ and $N = 1$ and

$$m_i = \sum_{j=0}^{i-1} 3^j = \frac{3^i - 1}{2}.$$

Theorem 4.1 *The set* $E = \{m : m \in \mathbf{Z}, \mathbf{N}'_0(m) = \hat{\mathbf{N}}'_0(m)\}$ *is the same as* $\{m_i : i \geq 0\}$.

PROOF. We show for all $m \geq 0$, that $m \in E$ if and only if $(m-1)/3 \in E$.

The leading coefficient of P_0 is 1, so $0 \in E$.

Let M' be the matrix for $U_{(0)}$ with respect to basis $\{3^{3m/2}d^m\}$.

Lemma 3.13 shows M'_{ij} has valuation at least $3i - 1$, so there is a matrix K over $Z[\sqrt{3}]$ and diagonal matrix $D = diag(3^{3i-1})$ such that $M' = DK$.

Let $\bar{K} = K \bmod \sqrt{3}\mathbf{Z}[\sqrt{3}]$ and let $c_m(\bar{K})$ be its upper $m \times m$ diagonal minor.

Every $m \times m$ diagonal minor of M' has valuation at least $\hat{\mathbf{N}}_0'(m)$ and the inquality is strict except for the upper $m \times m$ diagonal minor. So we have reduced the theorem to showing that $m \in E$ if and only if $c_m(\bar{K}) \neq 0$.

Call a degree m permutation π *excellent* if the selection of $\bar{K}$ associated to $(1, 2, \dots m)$ and π is a sequence of nonzero entries of $\bar{K}$.

Claim 1. If there is a degree m excellent π, then $m = m_i$ for some i.

We establish Claim 1 by induction. The trivial degree 0 permutation is excellent.

The entries of K satisfy a linear recurrence. Equation (12) with x substituted for δ is

$$G(x, y) = \frac{10xy + 4\sqrt{3}xy(x + 2y) + xy(x^2 + 2xy + 3y^2)}{1 - xy(10 + 4\sqrt{3}(x + y) + x^2 + xy + y^2)}.$$

The coefficient of $x^i y^j$ is the entry of K in row i and column j.

Let $\bar{G}$ be the generating function for entries of $\bar{K}$. $\bar{G}$ is the reduction of G to $\mathbf{F}_3[[x, y]]$.

Let

$$R(i) = (1 + (xy + x^3y + x^2y^2 + xy^3) + (xy + x^3y + x^2y^2 + xy^3)^2)^{3^i},$$

and

$$\bar{G}_0(x, y) = xy(1 - xy + y^2).$$

Let

$$\bar{G}_j = \bar{G}_0 \cdot \prod_{i=0}^{j-1} R(i)$$

and

$$\bar{C}_j = \prod_{i=j}^{\infty} R(i).$$

For all nonnegative integers j, $\bar{C}_j^3 = \bar{C}_{j+1}$ and $\bar{G} = \bar{G}_j \bar{C}_j$.

By direct computation,

$$\bar{G}_1 = (x^{-1}y + 1 - xy^{-1} + y^{-2})\bar{G}_0^3 + xy + x^2y^4 + x^6y^2, \tag{13}$$

and so

$$\bar{G} = (x^{-1}y + 1 - xy^{-1} + y^{-2})\bar{G}^3 + (xy + x^2y^4 + x^6y^2)\bar{C}_1. \tag{14}$$

Equation (14) shows the coefficient of $x^i y^{3j}$ in $\bar{G}$ is the same as the coefficient of $x^i y^{3j}$ in $\bar{G}^3$. This coefficient is zero if i is not divisible by 3.

Suppose degree m permutation π is excellent. The only unit in row 1 is in column 1, so $\pi(1) = 1$. The functions

$$\sigma(i) = \pi(3i)/3, \quad \sigma'(i) = (\pi(3i - 1) - 1)/3, \quad \sigma''(i) = (\pi(3i + 1) + 1)/3 \tag{15}$$

are excellent degree $\lfloor m/3 \rfloor$ permutations, and $3 \mid (m - 1)$.

The inductive step is complete.

Claim 2. For any m_i, there is a unique degree m_i excellent π.

We proceed by induction. The unique degree 0 permutation is excellent.

Equation (14) shows for excellent degree $\frac{m-1}{3}$ permutations $\sigma, \sigma', \sigma''$, there is an excellent degree m permutation π, computed by reversing Equations (15).

If there is a unique degree $(m-1)/3$ excellent σ, then there is a unique degree m excellent π. Claim 2 is established.

Claim 1 shows for m not equal to any m_i, that $c_m(\bar{K}) = 0$. Claim 2 shows for each m_i, there is a unique selection of the upper $m_i \times m_i$ diagonal major of $\bar{K}$ which contributes a nonzero term to $c_{m_i}(\bar{K})$. Hence, $c_m(\bar{K}) \neq 0$ if and only if there is i such that $m = m_i$. □

Corollary 4.1.1 *Let L be the secant line such that $L(m_i) = \hat{\mathbf{N}}'_0(m_i)$ and $L(m_{i+1}) = \hat{\mathbf{N}}'_0(m_{i+1})$. If m is such that $m_i < m < m_{i+1}$, then*

$$\hat{\mathbf{N}}'_0(m) < \mathbf{N}'_0(m) \leq L(m).$$

Proposition 4.2 *Let l be an integer, n be a nonnegative integer. Let $k = 2 \cdot 3^{n+1} \cdot l$. Let s be an integer, $0 \leq s < 2 \cdot 3^{n-1}$. If $\mathbf{N}'_0(s) = \hat{\mathbf{N}}'_0(s)$, then $\mathbf{N}'_k(s) = \hat{\mathbf{N}}'_0(s)$.*

Proof. Let $R = (S/V(S))^{k/3}$. The binomial theorem shows the coefficient of d_3^m in R has valuation at least $\lceil 3m/2 \rceil + n - v_3(m)$.

Let C be the matrix for the multiplication by R operator on $\mathcal{S}_0(1, \emptyset_p)$ with respect to the basis $\{3^{3m/2} d^m\}$.

Let M' be the matrix for $U_{(0)}$ with respect to the same basis.

By Theorem 1.5, $M'C$ is similar to a matrix for $U_{(k)}$.

For all i, j, $v_3(M'_{ij}) \geq 3i - 1$. For $i > 3j$ or $j > 3i$, $M'_{ij} = 0$.

For all $j > 0$, $C_{jj} = 1$. For $j, m > 0$, $v_3(C_{j+m,j}) \geq n - v_3(m)$ and $C_{j,j+m} = 0$.

For odd m, including $m = 3^{n-1}$, $v_3(C_{j+m,j}) \geq \frac{1}{2}$.

For all i, $v_3(M'_{ij} - (M'C)_{ij}) \geq 3i - 1$.

For $i \leq s$, $v_3(M'_{ij} - (M'C)_{ij}) > 3i - 1$, because

$$(M'C)_{ij} = \sum_{k=j}^{3i} M'_{ik} C_{kj},$$

and $3i \leq 3s < 2 \cdot 3^n$.

If $\mathbf{N}'_0(s) = \hat{\mathbf{N}}'_0(s)$ then $\mathbf{N}'_k(s) = \mathbf{N}'_0(s)$. □

Corollary 4.2.1 *Let l be an integer and n be a nonnegative integer. Let $k = 2 \cdot 3^{n+1} \cdot l$.*

For integer i, $0 \leq i < n-1$, there are exactly 3^i overconvergent 3-adic modular forms of weight k with slope in $[m_{i+1} + 1, m_{i+2} - 2]$, and these have average slope $3^{i+1} - 1$.

Proof. By Proposition 4.2, $\mathbf{N}'_k(m_i) = \mathbf{N}'_0(m_i)$ and $\mathbf{N}'_k(m_{i+1}) = \mathbf{N}'_0(m_{i+1})$. There are $3^i = m_{i+1} - m_i$ slopes with multiplicity accounted for by the edges joining these vertices of the Newton polygon $\mathbf{N}'_k$. The difference $\mathbf{N}'_k(m_{i+1}) - \mathbf{N}'_k(m_i)$ is $3^i(3^{i+1} - 1)$.

The average slope is $3^{i+1} - 1$. The minimum of these slopes is at least $3m_i + 2$ and the maximum at most $3m_{i+1} - 1$. □

Corollary 4.2.2 *Let k be an even integer and i be a positive integer. If*

$$v_3(k) \geq [\hat{\mathbf{N}}'_0(m_{i+1}) + \hat{\mathbf{N}}'_0(m_i)]/2 - \hat{\mathbf{N}}'_0((m_{i+1} + m_i)/2) + i + 2,$$

then for $m \leq m_{i+1}$, $\mathbf{N}'_0(m) = \mathbf{N}'_k(m)$.

Proof. The Newton polygons $\mathbf{N}'_0$ and $\mathbf{N}'_k$ both have vertices $(m_i, \hat{\mathbf{N}}'_0(m_i))$ and $(m_{i+1}, \hat{\mathbf{N}}'_0(m_{i+1}))$.

By Corollary 4.1.1 and Theorem 2.5, $v_3(a_m(P_k)) = v_3(a_m(P_0))$ for every m between m_i and m_{i+1}. □

Affirming a pattern noticed by Gouvêa[G],

Corollary 4.2.3 *Let $k = 2 \cdot 3^{n+1}$. The classical weight k level 3 oldforms have slopes outside $[k/4, 3k/4]$.*

Proof. There are $m_n = \frac{k}{12} - \frac{1}{2}$ cuspidal level 1 normalized eigenforms. There are $2m_i + 2$ classical level 3 oldforms, and one pair of these comes from the weight k Eisenstein series. The slopes of the forms in this pair are 0 and $k - 1$.

By Proposition 4.2, $\mathbf{N}'_k(m_n) = \hat{\mathbf{N}}'_0(m_n)$, because $m_n < 2 \cdot 3^{n-1}$.

The slope $\mathbf{N}'_k(m_n) - \mathbf{N}'_k(m_n - 1)$ is less than

$$\hat{\mathbf{N}}'_0(m_n) - \hat{\mathbf{N}}'_0(m_n - 1) = 3m_n - 1 = \frac{k}{4} - 1.$$

The mates of these m_i oldforms have slopes greater than $\frac{3k}{4}$. □

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